The Epistemology of Simulation

Poincaré's Conventionalism as a Proto-Theory of Virtual Worlds

Xuanlin Zhu

PhD Applicant: Computational Media / Critical Game Design
Portfolio Sample

Abstract

Before the advent of digital computation, Henri Poincaré proposed a thought experiment involving a "Heated Disk"—a world where spatial truths are dictated by variable environmental parameters (temperature) rather than fixed axioms. This monograph argues that Poincaré's model serves as an epistemological precursor to modern game engines. By contrasting Helmholtz's empiricism with Poincaré's conventionalism, I demonstrate that the "choice" of geometry in a virtual environment is not a matter of truth, but of design utility. I revisit Poincaré's derivation of hyperbolic geometry through the lens of the "physics engine," proposing that the convention is the fundamental unit of meaning in Critical Game Design.

I. Introduction: The Architect's Dilemma

The essence of geometry has historically occupied a precarious position between metaphysics and experimental science. For the game designer, however, this tension is practical: does the space of the game world exist *a priori* (hard-coded into the engine), or is it synthesized *a posteriori* by the player's movement through it?

Immanuel Kant famously argued that geometry is "synthetic a priori"—a fixed property of intuition.[1] For Kant, Euclidean geometry was not merely a mathematical system but the structure of spatial experience itself, independent of empirical observation yet universally applicable to the physical world. But the development of non-Euclidean geometry in the 19th century shattered this certainty. Nikolai Lobachevsky and Bernhard Riemann proved that the parallel postulate was independent of Euclid's other four axioms, demonstrating the existence of consistent geometric systems where multiple lines or no lines could be drawn parallel to a given line through an external point.

This discovery sparked a philosophical crisis. If multiple geometries were mathematically consistent, which one described the "true" structure of space? Two major responses emerged. Hermann von Helmholtz argued that geometry is learned through physical measurement—an empirical science determined by experience.[2] Henri Poincaré, by contrast, argued that geometry is a chosen "convention"—we select Euclidean geometry not because it is true, but because it is convenient.[3]

My central claim is that Poincaré's Conventionalism offers the most robust theoretical framework for understanding Virtual Reality and Game Design. When Poincaré describes a world where "temperature" distorts length, he is describing a vertex shader. When he argues that we "choose" Euclidean geometry for convenience, he is predicting the optimization shortcuts of modern rendering pipelines. By re-examining these geometric models, we can

[1] Immanuel Kant, *Critique of Pure Reason*, trans. Marcus Weigelt (London: Penguin Classics, 2003), B41.
[2] H. Helmholtz, "The Origin and Meaning of Geometrical Axioms," *Mind* 1, no. 3 (1876): 301–321.
[3] Henri Poincaré, *Science and Hypothesis* (Project Gutenberg, 1902), 77–79.

understand the game engine not just as software, but as a philosophical argument about the nature of space itself.

This essay proceeds in six parts. Sections II–IV present the three geometric models central to the Helmholtz–Poincaré debate: the Euclidean plane (Model A), the spherical surface (Model B), and the Heated Disk (Model C). I provide formal mathematical definitions and propositions for each model, demonstrating how creatures confined to these spaces would develop radically different geometries based on their measurements of distance and angle. Section V analyzes Helmholtz's empiricist conclusion that geometry is determined by experience. Section VI examines Poincaré's conventionalist counter-argument: that we can describe the same physical world using different geometries by adjusting our definitions of "rigid body" and "straight line." Section VII—the conclusion—reframes this debate as a theory of game engines, arguing that the modern game designer is the ultimate Conventionalist.

II. Model A: Helmholtz's Euclidean Plane

In Helmholtz's 1876 paper, he proposed a thought experiment considering a two-dimensional creature living on a plane. He analyzed what such a creature might perceive and how they would develop their geometry.

Helmholtz first considered such creatures living in a Euclidean Plane—what we will call Model A:

> Now if beings of this kind lived on an infinite plane, their geometry would be exactly the same as our planimetry. They would affirm that only one straight line is possible between two points, that through a third point lying without this line only one line can be drawn parallel to it, that the ends of a straight line never meet though it is produced to infinity, and so on.[4]

Helmholtz did not give clear definitions for concepts such as "distance" and "straight," taking the results of measurements for granted (a point Poincaré later challenged). Since Model A serves primarily as a baseline for comparison, we may assume it is a Real Euclidean Plane and summarize Helmholtz's assumptions as follows.

Proposition 2.1. *In Model A, only one line is possible between two points.*

This follows from Euclid's first postulate (I-1). A direct consequence is that if two lines intersect, they intersect at exactly one point.

Proposition 2.2. *In Model A, the shortest path between two points is part of the straight line between them.*

[4] Helmholtz, "The Origin and Meaning of Geometrical Axioms," 303.

This can be proven using calculus of variations, though it requires a precise definition of "measurement" of "distance."

Proposition 2.3. *In Model A, through a third point not on a given line, only one line can be drawn parallel to it.*

This is exactly the parallel postulate.

Proposition 2.4. *In Model A, the angle sum of any triangle is* $180°$*.*

This proposition is derived from the parallel postulate.

Proposition 2.5. *In Model A, there exist similar triangles that are not congruent to each other.*

For example, consider $O(0,0)$, $A(1,0)$, $B(0,1)$, $C(2,0)$, $D(0,2)$. Then $\triangle OAB \sim \triangle OCD$, but they are not congruent.

III. Model B: Helmholtz's Spherical Surface

Helmholtz then proceeds to describe a second model:

> But intelligent beings of the kind supposed might also live on the surface of a sphere. Their shortest or straightest line between two points would then be an arc of the great circle passing through them... Of parallel lines the sphere-dwellers would know nothing. They would declare that any two straightest lines, sufficiently produced, must finally cut not in one only but in two points. The sum of the angles of a triangle would be always greater than two right angles, increasing as the surface of the triangle grew greater.[5]

We formalize this as follows:

Definition 3.1. In Model B, a point is a point on the surface of a sphere.

Definition 3.2. In Model B, a straight line passing through two points is a great circle passing through the two given points.

Proposition 3.3. *In Model B, there is not always only one straight line between two points.*

For instance, there are infinitely many great circles that pass through two antipodes. A further consequence is that two lines intersect at two points (antipodes) instead of one.

Proposition 3.4. *In Model B, the shortest path between two points is an arc on the surface of a sphere, which is part of the great circle passing through the two given points.*

[5] Helmholtz, "The Origin and Meaning of Geometrical Axioms," 304.

Similar to Proposition 2.2, we can solve for the curve where the calculus of variation is zero on a sphere and obtain a great circle.

Proposition 3.5. *In Model B, there are no parallel lines.*

All great circles intersect with each other at two points.

Definition 3.6. In Model B, a triangle is the shape enclosed by three points and the three shortest paths connecting them.

Proposition 3.7. *In Model B, the angle sum of the triangle is always greater than* $180°$.

Proposition 3.8. *In Model B, the triangle with a greater area has a greater angle sum than the triangle with a smaller area.*

For triangle $\triangle ABC$, the area S can be solved by the following equation:

$$S_{\text{sphere}} + 4S = S_{L(\sphericalangle A)} + S_{L(\sphericalangle B)} + S_{L(\sphericalangle C)}$$

where $S_{\text{sphere}} = 4\pi R^2$ is the surface area of the sphere and $S_{L(\sphericalangle\theta)} = S_{\text{sphere}}\frac{(\sphericalangle\theta)^\circ}{180^\circ}$ is the surface area of the lune with corner angle $\sphericalangle\theta$.

Solving, we get:

$$S = R^2\pi\left(\frac{(\sphericalangle A)^\circ + (\sphericalangle B)^\circ + (\sphericalangle C)^\circ - 180^\circ}{180^\circ}\right)$$

Therefore a triangle with greater area has greater angle sum, and vice versa. Also, since the area cannot be zero or negative, the angle sum of any triangle is greater than $180°$.

Proposition 3.9. *In Model B, there are no two triangles that are similar but not congruent.*

Similar triangles in Model B must be congruent, since a triangle's area is determined by its three angles.

IV. Helmholtz's Empiricist Analysis

By raising Models A and B, Helmholtz made the following argument. Creatures living in Model A would develop a geometry similar to our Euclidean plane geometry, while creatures living in Model B would develop a totally different one. We have listed propositions that differ dramatically between the two models: the number of lines passing through two points, the number of intersection points of two intersecting lines, the number of parallel lines through a point, the angle sum of triangles, and the existence of similar triangles that are not congruent.

Helmholtz concluded that, without the ability to have an overview of their plane's properties (as we do when examining these models from higher-dimensional space), creatures can only develop geometry based on their own experience. According to Helmholtz, such experiences are all related to *measurements*—specifically, measurements of length and angles.

From the perspective of differential geometry, the difference between the two planes is their respective curvatures. Helmholtz quotes Gauss's *Theorema Egregium*, which states that the curvature of a surface can be determined by measuring angles and distances, even without knowing the exact shape from a higher dimension. Thus Helmholtz confirmed that the geometry a creature develops is determined by experience of measurements.

Disagreeing with Kant's claim that geometry is "a priori," Helmholtz's view is that geometry is "a posteriori"—determined by empirical experience. This position is summarized as *empiricism*.

V. Model C: Poincaré's Heated Disk

In 1902, Poincaré proposed the following thought experiment to support his idea of conventionalism:

> Suppose, for example, a world enclosed in a large sphere and subject to the following laws: The temperature is not uniform; it is greatest at the centre, and gradually decreases as we move towards the circumference of the sphere, where it is absolute zero. The law of this temperature is as follows: If R be the radius of the sphere, and r the distance of the point considered from the centre, the absolute temperature will be proportional to $R^2 - r^2$... If they construct a geometry, it will not be like ours, which is the study of the movements of our invariable solids; it will be the study of the changes of position which they will have thus distinguished, and will be "non-Euclidean displacements," and this will be non-Euclidean geometry. So that beings like ourselves, educated in such a world, will not have the same geometry as ours.[6]

Poincaré gives two major assumptions of physical laws concerning the ideas of a rigid body and the light of such a world. This model is later simplified and called Poincaré's disk. We may summarize it into a 2-dimensional disk as follows.

Definition 5.1. In Model C, points are points in the given disk with radius R. That is, let the center of the disk be $O(0,0)$; points are all points $P(x,y)$ such that $x^2 + y^2 < R^2$.

Definition 5.2. In Model C, at point $P(x,y)$, the distance of d on the disk would be measured to be $s = d\frac{R^2}{R^2-x^2-y^2}$ in the metric system.

[6] Poincaré, *Science and Hypothesis*, 77–79.

Another way to state this is that the **differential length element** of the disk is $\frac{R^2}{R^2-x^2-y^2}dv$, while the differential length element of Euclidean space is just dv. Then the length of some path C is $\int_{C(x,y)} \frac{R^2}{R^2-x^2-y^2}dv$.

Poincaré realized this setting by assuming Model C is a world with non-uniform temperature. The center of the disk has the highest temperature and the circumference has absolute zero temperature. For a point at distance r from the center, the temperature is $k(R^2 - r^2)$ for some constant k. He also assumed all objects have the same thermal expansion coefficient and that thermal equilibrium is reached instantly.

For example, a ruler with length 1 at the center would contract when moved closer to the circumference. At (x, y), it would contract to $\frac{R^2-x^2-y^2}{R^2}$; thus a distance of d on the disk would be measured to be $s = d\frac{R^2}{R^2-x^2-y^2}$ at this point.

Definition 5.3. In Model C, a "line" passing through two given points is an arc that is part of a circle whose ends are perpendicular to the circumference of the disk, or a diameter of the disk.

This is exactly the "P-line" from Poincaré disk model.

Poincaré realized this setting by assuming Model C is a world where, if a point has distance r from the center, the index of refraction at that point is inversely proportional to $R^2 - r^2$. A line is the path that light travels.

By constructing this model, Poincaré claimed that creatures in Model C would develop hyperbolic geometry to describe their world. We may check several properties of Model C.

Proposition 5.4. *In Model C, there is only one line passing through two given distinct points.*

Lemma 5.5. *The equation of a circle that is perpendicular to the disk is* $x^2 + y^2 + ax + by + R^2 = 0$.

Proof. The equation of a circle is $(x - x_0)^2 + (y - y_0)^2 = r_0^2$, with $C(x_0, y_0)$ as the center and r_0 as the radius.

Expanding: $x^2 + y^2 - 2x_0x - 2y_0y + x_0^2 + y_0^2 - r_0^2 = 0$.

Suppose it intersects the disk at some point P. By definition, $OP \perp PC$, so $\triangle OPC$ is a right triangle.

By the Pythagorean Theorem, $(\overline{OP})^2 + (\overline{CP})^2 = (\overline{CO})^2$.

So $R^2 + r_0^2 = x_0^2 + y_0^2$, thus $x_0^2 + y_0^2 - r_0^2 = R^2$.

Therefore the equation simplifies to $x^2 + y^2 - 2x_0x - 2y_0y + R^2 = 0$, which is in the form $x^2 + y^2 + ax + by + R^2 = 0$. □

Now, given two points $A(x_1, y_1)$ and $B(x_2, y_2)$, there is a unique solution for a and b. (For the case where A, O, B are collinear ($x_1y_2 = x_2y_1$), the solution is the diameter $x_1y = y_1x$.) Therefore there is only one line passing through two given distinct points.

Proposition 5.6. *In Model C, there are at least two lines parallel to a given line passing through a given point not on the given line.*

Suppose we have point P and line ℓ, where ℓ intersects the circumference at A and B. Then the rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are two limiting parallel rays.

Proposition 5.7. *In Model C, the space is infinite by the metric system.*

From the center of the disk to any point on the circumference, the total measured distance is:

$$\int_0^R \frac{R^2}{R^2 - r^2} dr = \infty$$

Proposition 5.8. *In Model C, the angle sum of a triangle is less than* 180°.

This follows from the axioms of hyperbolic geometry.

VI. Poincaré's Conventionalist Counter-Argument

After describing Model C, Poincaré made a striking observation. He argued that creatures living in this "Heated Disk" could describe their world using *either* hyperbolic geometry *or* Euclidean geometry—and both descriptions would be equally valid.

If they chose hyperbolic geometry, they would accept that:

- Light travels in curved paths (P-lines)
- Rulers contract as they move toward the edge
- The geometry of space itself is non-Euclidean

If they chose Euclidean geometry, they would instead claim that:

- Light travels in straight lines, but is refracted by a spatially-varying medium
- Rulers remain rigid, but forces contract all objects near the edge
- The geometry of space is Euclidean, but physical laws are complex

Poincaré's point is that *these are not competing truth claims*—they are competing *conventions*. We choose Euclidean geometry not because it is "true," but because it is most convenient given our experience. In Helmholtz's Model B (spherical surface), we could similarly describe the geometry as either spherical or Euclidean-with-corrections.

This is the core of Poincaré's *Conventionalism*: The question "What is the geometry of space?" is not a factual question but a choice of descriptive framework. We select the framework that makes our physical laws simplest.

VII. Conclusion: The Designer as Conventionalist

In 1902, Poincaré argued that if we lived in the "Heated Disk," we could describe our world using either Hyperbolic Geometry (accepting the curved light) or Euclidean Geometry (inventing "refraction laws" to explain the curves). We would choose the latter, he claimed, solely because it is "most convenient."

This is the exact dilemma of the modern Game Engine. A physics engine does not simulate the world; it simulates a *convention* of the world optimized for 60 frames per second. When we design a non-Euclidean game (like *Hyperbolica* or *Manifold Garden*), we are validating Poincaré's thesis: the "truth" of the space is secondary to the rules we impose upon it.

Consider the technical implementation. In Unity or Unreal Engine, the "world space" is defined by a coordinate system and a transformation matrix. When the player moves, we do not "discover" the geometry of the world—we *apply* a convention:

- Euclidean convention: Linear perspective projection, rigid body physics
- Hyperbolic convention: Curved ray-tracing, non-linear distance metrics
- Spherical convention: Great circle geodesics, positive curvature

Each convention produces a different phenomenology for the player. The "physics" is not a constraint we discover but a design choice we make.

Reichenbach and Grünbaum criticized Poincaré for emphasizing arbitrariness, arguing that "coordinative definitions" link geometry to physical facts. But in the context of Computational Media, Poincaré is vindicated. In a simulation, there are no "physical facts" to constrain us—only the conventions we write into the code. The game designer is the ultimate Conventionalist, proving that space itself is a medium of expression.

The Heated Disk is not merely a thought experiment—it is a description of every virtual world. Temperature becomes framerate; thermal expansion becomes level-of-detail culling; the index of refraction becomes shader complexity. When Poincaré wrote that geometry is "most convenient," he was predicting the optimization culture of modern game development, where "truth" is always secondary to performance.

This has implications beyond aesthetics. If we take Poincaré seriously, then Critical Game Design is not about "representing" reality but about *legislating* conventions of experience. The game designer, like Poincaré's hypothetical creatures, does not discover geometry—they *choose* it. And in that choice lies the expressive potential of the medium.

References

[1] Poincaré, Henri. *Science and Hypothesis*. Project Gutenberg, 1902.

[2] Helmholtz, H. "The Origin and Meaning of Geometrical Axioms." *Mind* 1, no. 3 (1876): 301–321.

[3] Kant, Immanuel. *Critique of Pure Reason*. Translated by Marcus Weigelt. London: Penguin Classics, 2003.

[4] Reichenbach, Hans. *The Philosophy of Space and Time*. Translated by Maria Reichenbach and John Freund. New York: Dover Publications, 1958.

[5] Grünbaum, Adolf. *Geometry and Chronometry in Philosophical Perspective*. Minneapolis: University of Minnesota Press, 1968.